A REVIEW OF OPTIMAL CONTROL SYSTEM ANALYSIS

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INTRODUCTION

Optimal control system analysis starts with the characterization of systems by state variables and the design of systems by state-space techniques. In general, optimal control problems are viewed as variational problems. There are many possible variational methods for maximizing or minimizing a functional over a function space. The range is from classical methods in the calculus of variations to numerical and successive approximation techniques of experimental or model systems. Among the commonly used methods in control system design are:

1. The calculus of variations.
2. The maximum principle.
3. Dynamic programming.

In all cases the goal is to find the optimum-control law or sequence such that a given function of performance indices is maximized or minimized. It will be found that the common property of all three methods is the use of variational principles. Each of these methods is related to well-known formulations in classical mechanics: the first to the Euler-Lagrange equation; the second to the Hamilton principle; and the third to the Hamilton-Jacobi theory. The maximum principle employs a more or less direct procedure of the calculus of variations, whereas dynamic programming, while still following the variational principles, uses the recurrence relationship or the algorithm of partial differential equations.

THE CALCULUS OF VARIATIONS

1. Outline

The calculus of variations is that branch of the calculus which is concerned with optimization problems under more general conditions than those considered in the ordinary theory of maxima and minima (extrema). There are three fundamental problems in the calculus of variations, the Lagrange problem, the Mayer problem, and the Bolza problem.

1) <u>The Lagrange Problem</u>. The Lagrange problem in one independent variable is concerned with the determination of a function $\underline{m}(t)$ which minimizes the integral of a given function. So the minimum integral control problem belongs to this type. It can be expressed in equations. Let the following be given:

a) A set of differential equations

$$\dot{x}_i = f_i(\underline{x}, \underline{m}, t), \text{ or more generally}$$

$$\varphi_i(\underline{x}, \dot{\underline{x}}, \underline{m}, t) = 0 \qquad (1\text{-}1)$$

$$i = 1, 2, \ldots, n$$

$\underline{x}$ is n-vector, $\underline{m}$ is r-vector.

b) A set of initial conditions

$$x_i(t_0) = a_i \qquad (1\text{-}2)$$

$$i = 1, 2, \ldots, n.$$

c) A criterion function

$$I = \int_{t_0}^{t_f} F(\underline{x}, \underline{m}, t)dt \qquad (1\text{-}3)$$

where $F(\underline{x}, \underline{m}, t)$ is a continuous function of the arguments.

Now the question is to determine the function $\underline{m}(t)$ which minimizes I over all functions of $\underline{m}(t)$, subject to the conditions given in equations (1-1) and (1-3).

2) The Mayer Problem. The Mayer problem is concerned with determination of a function $\underline{m}(t)$ which minimizes a given function evaluated at the end point, containing some variables whose final values are unspecified in advance. Usually time-optimal problems are classified as Mayer problems. This problem may be stated as follows:

a) Given a set of differential equations

$$\dot{x}_i = f_i(\underline{x}, \underline{m}, t)$$

or

$$\varphi_i \ (\underline{x}, \underline{\dot{x}}, \underline{m}, t) = 0 \qquad (1\text{-}4)$$

$$i = 1, 2, \ldots, n.$$

b) Given a set of initial conditions

$$x_i(t_0) = a_i \qquad (1\text{-}5)$$

$$i = 1, 2, \ldots n.$$

c) Given a set of final conditions

$$x_j(t_f) = b_j$$

where j belongs to some subset of the integers 1, 2, . . ., n, and t_f is unspecified.

d) Given a criterion function

$$I = G(\underline{x}, \underline{m}, t) \Big|_{t_0}^{t_f} \qquad (1\text{-}7)$$

Determine the function $\underline{m}(t)$ which minimizes I over all functions of $\underline{m}(t)$ subject to the conditions given in equations (1-4), (1-5), and (1-6).

3) <u>The Bolza Problem</u>. The Bolza problem is concerned with determination of a function $\underline{m}(t)$ which minimizes the integral of a function plus a function evaluated at the end point and contains some variables whose final values are unspecified in advance. An optimal control system subject to certain constraints can be studied as a problem of Bolza type. This is the most general case, and the Mayer problem and the Lagrange problem are the special cases of this type. This problem may be stated as follows:

a) Given a set of differential equations as described in equation (1-1), or (1-4).

b) Given a set of initial conditions as described in equation (1-2), or (1-5).

c) Given a set of finial conditions as described in equation (1-6).

d) Given a criterion function

$$I = G(\underline{x},\ \underline{m},\ t)\Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} F(\underline{x},\ \underline{m},\ t)\,dt\ . \tag{1-8}$$

Determine the function $\underline{m}(t)$ which minimizes I over all functions of $\underline{m}(t)$ subject to the conditions given in equations (1-1), (1-4), and (1-6).

It is easy to see that if $G(\underline{x},\ \underline{m},\ t)$ is equal to zero in equation (1-8), it becomes equation (1-3); if $F(\underline{x},\ \underline{m},\ t)$ is equal to zero in equation (1-8), it becomes equation (1-7). However, some auxiliary variables can always be introduced which transform a Lagrange problem into a Bolza problem or a Mayer problem, and vice versa. In other words, these problems can be

converted from one to another. Although there are many optimal control problems which do not seem to belong to any of these formulations, there is always some mathematical artifice which will reduce the initial scheme to one of those considered above.

2. Basic Principle of the Calculus of Variation in Minimization Problems

We consider the fixed-end-point system and the movable-end-point system.

1) Fixed-end-point System. Consider the problem of minimizing the integral

$$I = \int_{t_0}^{t_f} F(x, \dot{x}, t)dt \tag{1-9}$$

where $x = x(t)$ is a twice-differentiable function and satisfies the conditions $x(t_0) = x_0$ and $x(t_f) = x_f$, and F is a scalar continuous function of scalar arguments x, $\dot{x}$, t. Determine the function $x(t)$ which minimizes the integral of equation (1-9).

If one interprets this geometrically, the problem is to determine the curve $x(t)$ connecting the points (x_0, t_0) and (x_f, t_f) such that the integral along the curve of some given function $F(x, \dot{x}, t)$ is a minimum. In control terminology, if $x(t)$ is the output of a controlled system, then the integral given in equation (1-9) describes a measure of the overall performance of the system. The criterion of performance is that this integral is minimal. Let $x(t)$ be the minimizing function and $x(t)$ be a neighboring function of $x(t)$. The $x(t)$ and $x(t)$

are related by

$$\tilde{x}(t) = x(t) + \epsilon\eta(t) \tag{1-10}$$

$$\dot{\tilde{x}}(t) = \dot{x}(t) + \epsilon\dot{\eta}(t) \tag{1-11}$$

where ϵ is a small parameter, and $\eta(t)$ is an arbitrary differentiable function for which

$$\eta(t_0) = \eta(t_f) = 0 \ . \tag{1-12}$$

Since the end points are assumed to be fixed as shown in Fig. (1-1). The condition given in equation (1-12) insures that

$$\tilde{x}(t_0) = x(t_0) = x_0$$

and $$\tilde{x}(t_f) = x(t_f) = x_f \ .$$

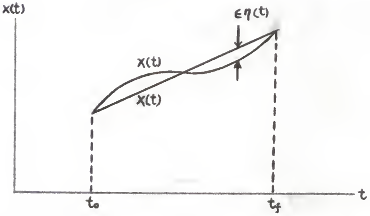

Fig. (1-1). Optimum trajectory with fixed-end points.

The vertical deviation of any curve $\tilde{x}(t)$ from the actual minimizing curve is given by $\epsilon\eta(t)$, as illustrated in Fig. (1-1). No matter which $\eta(t)$ is chosen, the minimizing function $x(t)$ is a member of that family for the choice of parameter value $\epsilon = 0$.

Replacing x and $\dot{x}$ in equation (1-9), respectively, by $\tilde{x}$ and $\dot{\tilde{x}}$ yields

$$I(\epsilon) = \int_{t_0}^{t_f} F(x + \epsilon\eta \ , \ \dot{x} + \epsilon\dot{\eta} \ , \ t)dt \tag{1-13}$$

By Taylor's series expansion, equation (1-13) becomes

$$I(\epsilon) = \int_{t_0}^{t_f} \Big[F(x,\dot{x},t) + \epsilon\eta\frac{\partial F}{\partial x} + \epsilon\dot{\eta}\frac{\partial F}{\partial \dot{x}} + \frac{1}{2!}\,(\epsilon^2\eta^2\frac{\partial^2 F}{\partial x^2}$$

$$+ \epsilon^2\,\eta\dot{\eta}\frac{\partial^2 F}{\partial x \partial \dot{x}} + \epsilon^2\,\dot{\eta}^2\,\frac{\partial^2 F}{\partial \dot{x}^2}) \Big] dt \tag{1-14}$$

The necessary condition for I to be a maximum or a minimum is that

$$\left.\frac{\partial I(\epsilon)}{\partial \epsilon}\right|_{\epsilon = 0} = 0 \ . \tag{1-15}$$

This leads to the condition that

$$\int_{t_0}^{t_f} (\eta\frac{\partial F}{\partial x} + \dot{\eta}\frac{\partial F}{\partial \dot{x}})dt = 0 \tag{1-16}$$

Equation (1-16) is obtained by omitting the terms ϵ^2, ϵ^3, . . ., in equation (1-14).

Integrating by parts the second term of this integral

$$\int_{t_0}^{t_f} \dot{\eta}\,\frac{\partial F}{\partial \dot{x}}\,dt = \left.\eta\frac{\partial F}{\partial \dot{x}}\right|_{t_0}^{t_f} - \int_{t_0}^{t_f} \frac{d}{dt}(\frac{\partial F}{\partial \dot{x}})dt$$

$$= - \int_{t_0}^{t_f} \eta\,\frac{d}{dt}(\frac{\partial F}{\partial \dot{x}})dt \tag{1-17}$$

$\left.\eta\frac{\partial F}{\partial \dot{x}}\right|_{t_0}^{t_f} = 0$ by the condition of equation (1-12).

The equation (1-16) reduces to

$$\int_{t_0}^{t_f} \eta \left[\frac{\partial F}{\partial x} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{x}}\right) \right] dt = 0 \tag{1-18}$$

Since equation (1-18) must hold for all η , the necessary condition for I to be an extremum is

$$\frac{\partial F}{\partial x} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}} = 0 \tag{1-19}$$

This second-order equation is known as the Euler-Lagrange differential equation and its solution gives the minimizing function on the integral of the problem provided the minimum exists.

In the case of multidimensional functions, the criterion integral of equation (1-9) becomes

$$I = \int_{t_0}^{t_f} F(\underline{x}, \dot{\underline{x}}, t)dt \tag{1-20}$$

where $\underline{x} = \underline{x}(t)$ is an n-vector function of t,each component function being twice differentiable.

$$\underline{x}(t) = \left[x_1(t), x_2(t), \ . \ . \ ., x_n(t)\right]' \tag{1-21}$$

The Euler-Lagrange differential equation for the multidimensional case is

$$\nabla_x F - \frac{d}{dt}(\nabla_{\dot{x}} F) = 0 \tag{1-22}$$

where $$\nabla_x F = \left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \ . \ . \ ., \frac{\partial F}{\partial x_n}\right)' \tag{-23}$$

and

$$\nabla_{\dot{x}} F = \left(\frac{\partial F}{\partial \dot{x}_1}, \frac{\partial F}{\partial \dot{x}_2}, \ . \ . \ ., \frac{\partial F}{\partial \dot{x}_n}\right)' \tag{1-24}$$

2) Movable-end-point system. Now we discuss the minimization problem with the end point of the trajectory lying on a curve $x = c(t)$, as shown in Fig. (1-2).

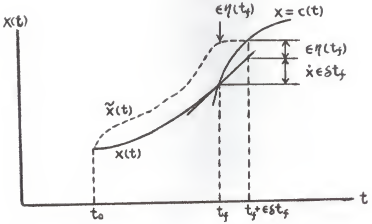

Fig. (1-2). Optimal trajectory with movable-end-point.

Let $x(t)$ be the function which minimizes the criterion integral given in equation (1-9). The end point of the trajectory is assumed to lie on the curve $x = c(t)$, as shown in Fig. 1-2). Assume that $x(t)$ is a neighboring function of $x(t)$. The relationship between $x(t)$ and $x(t)$ is the same as equations (1-10) and (1-11). The arbitrary function $\eta(t)$ satisfies the initial condition

$$\eta(t_0) = 0 \tag{1-25}$$

but the final condition is yet undefined.

By replacing x and $\dot{x}$ in equation (1-9), respectively, by $\tilde{x}$ and $\dot{\tilde{x}}$, and the upper limit of integration by $t_f + \epsilon\delta t_f$, one obtains

$$I(\epsilon) = \int_{t_0}^{t_f+\epsilon\delta t_f} F(x + \epsilon\eta,\ \dot{x} + \epsilon\dot{\eta}\ ,\ t)dt$$

$$= \int_{t_0}^{t_f} F(x+ \epsilon\eta,\ \dot{x}+ \epsilon\dot{\eta}\ ,t)dt + \int_{t_0}^{t_f+\epsilon\delta t_f} F(x+ \epsilon\eta\ ,\dot{x}+ \epsilon\dot{\eta}\ ,t)dt$$

$$= \int_{t_0}^{t_f} (\epsilon\eta \frac{\partial F}{\partial x} + \epsilon\dot{\eta}\ \frac{\partial F}{\partial \dot{x}})dt + \epsilon\,\delta t_f$$

$$\cdot\ F\left[x(\xi) + \epsilon\eta(\xi),\ \dot{x}(\xi) + \epsilon(\xi),\ \xi\right]dt$$

$$= \epsilon\int_{t_0}^{t_f} (\eta\frac{\partial F}{\partial x} + \dot{\eta}\ \frac{\partial F}{\partial x})dt + \epsilon\,\delta t_f$$

$$\cdot\ F\left[x(t_f) + \epsilon\eta(t_f),\ \dot{x}(t_f) + \epsilon\dot{\eta}(t_f),\ t_f\right]dt$$

$$= \epsilon\int_{t_0}^{t_f} (\eta\frac{\partial F}{\partial x} + \dot{\eta}\ \frac{\partial F}{\partial \dot{x}})dt + \epsilon\,\delta t_f F(t_f) \qquad (1\text{-}26)$$

where $t_f \leqq \xi \leqq t_f + \epsilon\delta t_f$.

As $\epsilon\delta t_f$ is very small, we can replace ξ by t_f. The necessary condition for I to be an extremum is

$$\frac{\partial I(\epsilon)}{\partial \epsilon} = 0$$

Therefore

$$\int_{t_0}^{t_f} (\eta\frac{\partial F}{\partial x} + \dot{\eta}\ \frac{\partial F}{\partial \dot{x}})dt + \delta t_f F(t_f) = 0 \qquad (1\text{-}27)$$

Integrating the term $\dot{\eta}\,\frac{\partial F}{\partial \dot{x}}$ as before, and rearranging

$$\int_{t_0}^{t_f} \eta\left[\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)\right] dt + \eta(t_f)\frac{\partial F}{\partial \dot{x}}\bigg|_{t_f} - \eta(t_0)\frac{F}{\dot{x}}\bigg|_{t_0}$$

$$+ F(t_f)\,\delta t_f = 0 \qquad (1\text{-}28)$$

Examination of the end condition shown in Fig. (1-2) leads to the following relation:

$$\dot{x}\epsilon\delta t_f + \epsilon\eta(t_f) = \dot{c}(t_f)\epsilon\,\delta t_f$$

or
$$\dot{x}\delta t_f + \eta(t_f) = \dot{c}(t_f)\,\delta t_f \qquad (1\text{-}29)$$

Substituting equation (1-29) into equation (1-28) yields

$$\int_{t_0}^{t_f} \eta\left(\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)\right)dt + \left\{F(t_f) + \left[\dot{c}(t_f) - \dot{x}(t_f)\right]\frac{F}{\dot{x}}\bigg|_{t_f}\right\}\delta t_f$$

$$- \eta(t_0)\frac{\partial F}{\partial \dot{x}}\bigg|_{t_0} = 0 \qquad (1\text{-}30)$$

From boundary condition $\eta(t_0) = 0$ and δt_f is arbitrary, equation (1-30) leads to

$$\int_{t_0}^{t_f} \eta\left[\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)\right] dt = 0 \qquad (1\text{-}31)$$

and

$$\left\{F(t_f) + \left[\dot{c}(t_f) - \dot{x}(t_f)\,\frac{F}{\dot{x}}\bigg|_{t_f}\right]\right\} = 0 \qquad (1\text{-}32)$$

Since $\eta \neq 0$ in equation (1-31), it follows that

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 \qquad (1\text{-}33)$$

This is exactly the same form discussed in case 1, the fixed-end-point system.

From equation (1-32)

$$F(t_f) = \left[\dot{x}(t_f) - \dot{c}(t_f)\right] \frac{\partial F}{\partial \dot{x}}\bigg|_{t_f} \tag{1-34}$$

For a special case, if the end point curve $x = c(t)$ is a horizontal straight line and

$$\dot{c}(t_f) = 0$$

equation (1-34) becomes

$$F(t_f) = \dot{x}(t_f) \frac{\partial F}{\partial \dot{x}}\bigg|_{t_f} \tag{1-35}$$

For movable-end-point system, the conditions of equations (1-33) and (1-34) must be hold, and equation (1-34) is referred to as the transversality condition.

3. Application of the Calculus of Variation

Minimum-integral control problems are to be studied as an application of the variational calculus to the optimum design of control processes. Consider a control characterized by

$$\dot{x}(t) = g(x, m) \tag{1-36}$$

where x and m are analytic scalar functions of t, and at $t = t_0$

$$x(t_0) = x_0 \tag{1-37}$$

Determine the optimum control signal m(t) which minimizes the integral-criterion function

$$I = \int_{t_0}^{t_f} F(x, m)\,dt \tag{1-38}$$

Let x and m be a pair of functions which yield the minimum of equation (1-38). Then the neighboring functions $\tilde{x}$ and $\tilde{m}$ may be

expressed as

$$\tilde{x} = x + \epsilon\eta \tag{1-39}$$

$$\tilde{m} = m + \epsilon\xi \tag{1-40}$$

where η and ξ are arbitrary differentiable functions of t, defined for $t_0 \leq t \leq t_f$, and ϵ is a small parameter. Replacing x and m in equation (1-38), respectively, by x and m, yields the neighboring function

$$I\ (\epsilon) = \int_{t_0}^{t_f} F(x + \epsilon\eta\ ,\ m + \epsilon\xi\ )dt \tag{1-41}$$

By Taylor's series expansion, one obtains

$$I(\epsilon) = \int_{t_0}^{t_f} F(x,\ m)dt + \epsilon\int_{t_0}^{t_f} (\eta\frac{\partial F}{\partial x} + \xi\frac{\partial F}{\partial m})dt$$

$$+ \text{ terms } \epsilon^2,\ \epsilon^3,\ .\ .\ . \tag{1-42}$$

We omit the high-order infinitesimal terms $\epsilon^2, \epsilon^3, . . .$ since the minimum of the integral occurs when

$$\left.\frac{\partial I}{\partial \epsilon}\right|_{\epsilon=0} = 0 \tag{1-43}$$

The necessary condition for I to be an extremum is

$$\int_{t_0}^{t_f} (\eta\frac{\partial F}{\partial x} + \xi\frac{\partial F}{\partial m})dt = 0 \tag{1-44}$$

Replacing x and m in equation (1-36), respectively, by $\tilde{x}$ and $\tilde{m}$ yields

$$\dot{x} + \epsilon\dot{\eta} = g(x + \epsilon\eta\ ,\ m + \epsilon\xi\ ) \tag{1-45}$$

By Taylor's series expansion

$$\dot{x} + \epsilon\dot{\eta} = g(x,m) + \epsilon\left(\eta\frac{\partial g}{\partial x} + \xi\frac{\partial g}{\partial m}\right) + \text{terms } \epsilon^2, \epsilon^3, \ldots \quad (1\text{-}46)$$

From equation (1-46), we know

$$\dot{x} = g(x, m) \quad (1\text{-}47)$$

$$\dot{\eta} = \eta\frac{\partial g}{\partial x} + \xi\frac{\partial g}{\partial m} \quad (1\text{-}48)$$

Solving for from equation (1-48), we obtain

$$\xi = \frac{\dot{\eta} - \eta\,(\partial g/\partial x)}{\partial g/\partial m} \;. \quad (1\text{-}49)$$

Substituting equation (1-49) into equation (1-44) yields

$$\int_{t_0}^{t_f} \eta\frac{\partial F}{\partial x} + \left(\dot{\eta} - \eta\frac{\partial g}{\partial m}\right)\frac{\partial F/\partial m}{\partial g/\partial m}\, dt = 0 \quad (1\text{-}50)$$

Integrating by parts yields

$$\int_{t_0}^{t_f} \frac{\partial F/\partial m}{\partial g/\partial m}\,\dot{\eta}\, dt = \eta\frac{\partial F/\partial m}{\partial g/\partial m}\Bigg|_{t_0}^{t_f} - \int_{t_0}^{t_f} \eta\frac{d}{dt}\left(\frac{\partial F/\partial m}{\partial g/\partial m}\right) dt$$

$$= -\int_{t_0}^{t_f} \eta\frac{d}{dt}\left(\frac{\partial F/\partial m}{\partial g/\partial m}\right) dt \quad (1\text{-}51)$$

Since $x(t_0) = x_0$, $\eta(t_0) = 0$ and

$$\frac{\partial F/\partial m}{\partial g/\partial m}\Bigg|_{t_f} = 0 \quad (1\text{-}52)$$

Equation (1-50) becomes

$$\int_{t_0}^{t_f} \eta\frac{\partial F}{\partial x}\, dt - \int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{\partial F/\partial m}{\partial g/\partial m}\right) dt - \int_{t_0}^{t_f} \eta\frac{\partial g}{\partial x}\,\frac{\partial F/\partial m}{\partial g/\partial m}\, dt = 0$$

and can be rewritten as

$$\int_{t_0}^{t_f}\left[\frac{\frac{\partial F}{\partial x}\cdot\frac{\partial g}{\partial m}}{\partial g/\partial m} - \frac{\partial g}{\partial x}\ \frac{\partial F/\partial m}{\partial g/\partial m}\right]\eta\ dt - \int_{t_0}^{t_f}\eta\ \frac{d}{dt}\left(\frac{\partial F/\partial m}{\partial g/\partial m}\right)dt = 0$$

and furthermore, as

$$\int_{t_0}^{t_f}\left[\frac{\frac{\partial F}{\partial x}\ \frac{\partial g}{\partial m} - \frac{\partial F}{\partial m}\ \frac{\partial g}{\partial x}}{\partial g/\partial m} - \frac{d}{dt}\left(\frac{\partial F/\partial m}{\partial g/\partial m}\right)\right]\eta\ dt = 0 \qquad (1\text{-}53)$$

Since equation (1-53) must hold for all η, the Euler-Lagrange differential equation for this optimum control problem evolves as

$$\frac{\frac{\partial F}{\partial x}\ \frac{\partial g}{\partial m} - \frac{\partial F}{\partial m}\ \frac{\partial g}{\partial x}}{\partial g/\partial m} - \frac{d}{dt}\left(\frac{\partial F/\partial m}{\partial g/\partial m}\right) = 0 \qquad (1\text{-}54)$$

The solution of this differential equation subject to the boundary conditions specified in equations (1-37) and (1-52) gives the optimum control m(t) for the process.

For a special case

$$\dot{x} = g(x,\ m) = m$$

the integral criterion function becomes

$$I = \int_{t_0}^{t_f} F(x,\ \dot{x})dt$$

and $$\frac{\partial g}{\partial m} = 1, \quad \frac{\partial g}{\partial x} = 0.$$

The Euler-Lagrange equation for this special case reduces to

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial x}\right) = 0$$

or
$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0$$

with the boundary conditions

$$x(t_0) = x_0 \text{ and } \left.\frac{\partial F}{\partial \dot{x}}\right|_{t_f} = 0$$

This is the same result as equation (1-33).

The same process extends to a multidimensional control process by replacement of the scalar function by a multidimensional vector.

Previous discussions assume that both control signals and state variables are subject to no constraints. But in realistic optimum controls for physical processes, constraints on control signals or state variables must be taken into consideration. For solving those optimum control problems with constraints, we usually use the method of Lagrange multipliers which will be discussed in the section on dynamic programming.

The optimum design of control systems by the calculus of variations leads to a two-point boundary value problem. Analytical solutions for such problems are possible only in special cases. So trial-and-error techniques must be resorted to. These techniques first guess a value for the missing initial condition, and integrate numerically the Euler-Lagrange and constraining equations. There must be a difference between the resulting final condition and the specified condition. So the trial-and-error process must be repeated several times until the value of the final condition obtained in this way agrees sufficiently

close with the specified value of the final condition. Because of this shortcoming, the classical calculus of variations is less attractive in the design of optimal control systems.

THE MAXIMUM PRINCIPLE

1. Outline

The maximum principle of Pontryagin is generally regarded as most promising for the solution of complex problems. The Russian mathematician Pontryagin discovered that the control problem can precede the calculus of variations by relating the Pontryagin function and the Hamiltonian function. In fact, Pontryagin's maximum principle bears a close relation to the classical problem of Mayer. It differs, however, from the Mayer problem in one respect. In the Mayer problem, every control signal is unbounded. In Pontryagin's work, on the other hand, the values of the control signal are bounded. A control vector which satisfies the constraint conditions is referred to as an admissible control vector.

As in the calculus of variations, Pontryagin's maximum principle can be used on three basic problems.

1) The Minimum-time Control Problem. The minimum-time control problem may be stated as the determination of an admissible control vector $\underline{m}$ so that the process is taken from a specified initial state vector $\underline{x}_0$ to a desired final state vector $\underline{x}_f$ in the shortest possible time.

2) The Terminal-control Problem. The terminal-control problem is the determination of an admissible control vector $\underline{m}$ such that, in a given time interval T, the system is taken from an initial state $\underline{x}_0$ into a state in which one or a combination of the state variables becomes as large as possible or as small as possible, and the remaining state variables have fixed values within physical limits.

3) The Minimum-integral Control Problem. The minimum-integral control problem may be stated as the determination of an admissible control vector $\underline{m}$ in such a manner that the integral

$$I = \int_{t_0}^{t_f} F(\underline{x}, \underline{m}, t)dt$$

reduces to a minimum during the time of movement $t_f - t_0$.

These three modes of optimum control can be transformed to an optimization with respect to co-ordinates or state variables, which is referred to as "generalized mode of optimum control". The transformation is carried out by "invariant embedding", a procedure of increasing the dimensionality of the state vector by adding a new co-ordinate. The reduction of the three modes of optimum control to the generalized mode is presented as follows.

4) Reducing a Minimum-time Control Problem to the Generalized Mode of Optimal Control. Consider the n^{th}-order control process characterized by

$$\dot{x}_i = f_i(\underline{x}, \underline{m}, t) \qquad (2\text{-}1)$$

$$i = 1, 2, \ldots, n$$

This problem implies the minimization of the time required to move the process from an initial state to a desired final

state, i.e.,

$$\min_{\underline{m}} \int_{t_0}^{t} d\tau \tag{2-2}$$

By introducing a new state variable $x_{n+1}(t)$ such that

$$\dot{x}_{n+1}(t) = 1 \tag{2-3}$$

$$x_{n+1}(t) = \int_{t_0}^{t} d\tau \tag{2-4}$$

The optimal control problem reduces to determination of an admissible control vector $\underline{m}$ so that the new state variable x_{n+1} is minimized.

5) Reducing a Terminal-control Problem to the Generalized Mode of Optimal Control. Consider the n^{th}-order control process characterized by equation (2-1). Now introduce a new state variable

$$x_{n+1}(t) = F\left[x_1(t),\ x_2(t),\ .\ .\ .,\ x_n(t)\right] \tag{2-5}$$

with the initial value given by

$$x_{n+1}(0) = F\left[x_1(0),\ x_2(0),\ .\ .\ .,\ x_n(0)\right] \tag{2-6}$$

In vector notation

$$x_{n+1}(t) = F\left[\underline{x}(t)\right] \tag{2-7}$$

$$x_{n+1}(0) = F\left[\underline{x}(0)\right] \tag{2-8}$$

$$\dot{x}_{n+1}(t) = \sum_{k=1}^{n} \frac{\partial F(\underline{x})}{\partial x_k} f_k(\underline{x},\ \underline{m},\ t) \tag{2-9}$$

Derivatives of the other state variables are given by equation (2-1). The terminal control problem is now reduced to the problem of optimization with respect to the new co-ordinate x_{n+1}, at the final moment of time.

6) Reducing a Minimum-integral Control Problem to the Generalized Mode of Optimal Control. Consider the n^{th}-order control

process characterized by equation (2-1). By introducing a new state variable $x_{n+1}(t)$ defined by

$$x_{n+1}(t) = \int_{t_0}^{t} F(\underline{x}, \underline{m}, t)dt \tag{2-10}$$

$$x_{n+1}(t_0) = 0 \tag{2-11}$$

$$\dot{x}_{n+1}(t) = F(\underline{x}, \underline{m}, t) \tag{2-12}$$

Now the problem becomes the problem of minimizing the $(n+1)^{th}$ co-ordinate, $x_{n+1}(t)$, at the terminal of the trajectory $t = t_f$.

The optimal control problem discussed above may be considered as a special case of the more general problem of maximizing or minimizing the Pontryagin function

$$\mathcal{P} = \sum_{i=1}^{n} b_i x_i(t_f) \tag{2-13}$$

This can be written in a vector form

$$\mathcal{P} = \left(\underline{b}, \underline{x}(t_f)\right) = \underline{b}'\underline{x}(t_f) \tag{2-14}$$

where $\underline{x}$ is a state vector of n^{th}-order control process under consideration, and $\underline{b}$ is a column vector which depends upon the co-ordinates to be minimized or maximized. A simple geometrical interpretation of the maximum principle is that the control vector $\underline{m}$ is chosen in such a way that the state vector $\underline{x}(t_f)$ moves "farthest" in the direction of "$-\underline{b}$", and thus the Pontryagin function $\mathcal{P}$ takes on a minimum value. In optimal control problems the final state of optimal trajectory may be either free or constrained. Equation (2-14) is unconstrained. If the final state of process is constrained by

$$\underline{R}_k \left[\underline{x}(t_f)\right] = 0 \qquad (2\text{-}15)$$

$$k = 1, 2, \ldots, r$$

the Pontryagin function takes the form

$$\Phi = \underline{b}'\underline{x}(t_f) + \underline{u}'\underline{R}\ \underline{x}(t_f) \qquad (2\text{-}16)$$

where $\underline{u}$ is a vector Lagrange multiplier.

Frequently, the extremization of the Pontryagin function is not easy to accomplish. Pontryagin first discovered the simplicity of the Hamiltonian function and its very nature makes it tempting to think that maximization of the Pontryagin function, and the use of the Hamiltonian, may lead simply to elegant methods for solving optimization problems.

Now, the maximum (or minimum) principle states that, if the control vector $\underline{m}$ is optimum, i.e., if it minimizes (or maximizes) the Pontryagin function Φ, then the Hamiltonian $H(\underline{x}, \underline{p}, \underline{m}, t)$ is maximized (or minimized) with respect to $\underline{m}$ over the control interval. The Hamiltonian is defined as

$$H(\underline{x}, \underline{p}, \underline{m}, t) = \sum_{j=1}^{n} P_j f_j \qquad (2\text{-}17)$$

where $\underline{x}$ is the state vector, $\underline{P}$ is the momentum vector defined as the solution to the differential equation

$$\dot{p}_i = - \sum_{j=1}^{n} p_j \frac{\partial f_i}{\partial x_i} \qquad (2\text{-}18)$$

where $p_i(t_f) = -b_i, \quad i = 1, 2, \ldots, n \qquad (2\text{-}19)$

b_i being some known constant specified in the Pontryagin function in equation (2-13).

If we differentiate equation (2-17) with respect to p_i

$$\frac{\partial H}{\partial P_i} = f_i(\underline{x}, \underline{m}, t) \tag{2-20}$$

Differentiate equation (2-17) with respect to x_i

$$\frac{\partial H}{\partial x_i} = \sum_{j=1}^{n} P_i \frac{\partial f_i}{\partial x_i} \tag{2-21}$$

$$i = 1, 2, \ldots, n$$

Comparing equations (2-1), (2-18) with equations (2-20), (2-21) we obtain the Hamiltonian canonical form

$$\dot{x}_i = \frac{\partial H}{\partial P_i} \tag{2-22}$$

$$\dot{P}_i = -\frac{\partial H}{\partial x_i} \tag{2-23}$$

These canonical equations are subject to the boundary condition on $x_i(t_0)$ and $p_i(t_f)$; that is,

$$x_i(t_0) = x_i^0 \tag{2-24}$$

and

$$p_i(t_f) = -b_i \tag{2-25}$$

$$i = 1, 2, \ldots, n$$

The physical interpretation of the maximum principle may be stated that the Hamiltonian H is the inner product of $\underline{P}$ and $\underline{f}$, or that of $\underline{P}$ and $\dot{\underline{x}}$, which represents the power when $\underline{P}$ is identified as the momentum. Thus to minimize $\wp$, the power is maximized, and when $\wp$ is minimum, H is a maximum.

2. Proof of the Maximum Principle

Proof is initiated with determination of a variation of the Pontryagin function, δ , due to a variation of state variation, δx_i, and a change in control signal, $\delta \underline{m}$. Assume that the n^{th}-order control process is characterized by

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{m}, t) \tag{2-26}$$

where $\underline{x}$ is n-state vector, and $\underline{m}$ is r-control vector. The variation $\delta\mathcal{P}$ as a function of $\delta\underline{m}$ and δx_i is first derived. To begin with the following summation is formed:

$$\sum_{i=1}^{n} P_i \, \delta x_i \tag{2-27}$$

Taking the derivative of this summation with respect to t

$$\frac{d}{dt} \sum_{i=1}^{n} P_i \delta x_i = \sum_{i=1}^{n} P_i \delta \dot{x}_i + \sum_{i=1}^{n} \dot{P}_i \delta x_i \tag{2-28}$$

Integrating both sides of equation (2-28) from t_0 to t_f, and simplification leads to

$$\sum_{i=1}^{n} P_i \delta x_i \Big|_{t_0}^{t_f} = \int_{t_0}^{t_f} \sum_i P_i \left[f_i(\underline{x}+\delta\underline{x}, \underline{m}+\delta\underline{m}, t) - f_i(\underline{x},\underline{m},t) \right] dt + \int_{t_0}^{t_f} \sum_i \dot{P}_i \delta x_i \tag{2-29}$$

Since $p_i(t_f) = -b_i$ and $x_i(t_0) = 0$. The left side of equation (2-29)

$$\sum_i P_i \delta x_i \Big|_{t_0}^{t_f} = - \sum b_i \delta x_i(t_f) = -\delta\mathcal{P} \tag{2-30}$$

Thus the variation of the Pontryagin function $\mathcal{P}$ due to change in x_i and $\underline{m}$ is

$$\delta \quad = -\int_{t_0}^{t_f} \sum_i P_i \left[f_i(\underline{x}+\delta\underline{x},\ \underline{m}+\delta\underline{m},\ t) - f_i(\underline{x},\underline{m},t)\right] dt$$

$$- \int_{t_0}^{t_f} \sum_i \dot{P}_i \delta x_i dt \qquad (2\text{-}31)$$

A Taylor series expansion of the integrand with respect to $\underline{x}$ is executed. Omitting the higher order terms and using the relationship of equations (2-18) and (2-23), and equation (2-31), one obtains

$$\delta\wp = \int_{t_0}^{t_f} \sum_i \sum_j P_i \frac{\partial f_i(\underline{x},\underline{m},t)}{\partial x_i} \delta x_i dt$$

$$- \int_{t_0}^{t_f} \sum_i P_i \left\{ f_i(\underline{x},\underline{m}+\delta\underline{m},\ t) - f_i(\underline{x},\underline{m},t) \right.$$

$$+ \sum_j \frac{\partial f_i(\underline{x},\ \underline{m}+\delta\underline{m},\ t)}{\partial x_j} \delta x_j$$

$$\left. + \frac{1}{2} \sum_j \sum_k \frac{\partial^2 f_j(\underline{x}+\theta\delta\underline{x},\ \underline{m}+\delta\underline{m},\ t)}{\partial x_j\, \partial x_k} \partial x_j\, \partial x_k \right\} dt \qquad (2\text{-}32)$$

Rearrangement yields

$$\delta\wp = - \int_{t_0}^{t_f} \sum P_i \left\{ f_i(\underline{x},\ \underline{m}+\delta\underline{m},\ t) - f_i(\underline{x},\underline{m},t) \right\} dt$$

$$- \int_{t_0}^{t_f} \sum \sum P_i \left\{ \frac{\partial f_i(\underline{x},\ \underline{m}+\delta\underline{m},\ t)}{\partial x_j} - \frac{\partial f_i(x,m,t)}{\partial x_j} \right\} \delta x_j dt$$

$$- \frac{1}{2} \int_{t_0}^{t_f} \sum_i \sum_j \sum_k P_i \frac{\partial^2 f_i(\underline{x}+\theta\delta\underline{x},\ \underline{m}+\delta\underline{m},\ t)}{\partial x_j\, \partial x_k} \delta x_j \delta x_k dt \qquad (2\text{-}33)$$

In view of equation (2-17) and defining R as

$$R = \int_{t_0}^{t_f} \sum_i \sum_j P_i \left\{ \frac{\partial [f_i(\underline{x}, \underline{m}+\delta\underline{m}, t) - f_i(\underline{x},\underline{m},t)]}{\partial x_j} \right\} \delta x_j dt$$

$$+ \frac{1}{2} \int_{t_0}^{t_f} \sum_i \sum_j \sum_k P_i \frac{\partial^2 f_i(x+\theta\delta x, m+\delta m, t)}{\partial x_j \, \partial x_k} \delta x_j \delta x_k dt \tag{2-34}$$

$\delta\wp$ may be expressed as

$$\delta\wp = - \int_{t_0}^{t_f} \left[H(\underline{x},\underline{P},\underline{m}+\delta\underline{m},t) - H(\underline{x},\underline{P},\underline{m},t) \right] dt - R \tag{2-35}$$

First, we want to show the necessary condition which states that if the Hamiltonian H is not a maximum, the minimum condition for $\wp$ is violated. The condition for the Pontryagin function $\wp$ to be the minimum for any small change $\delta\underline{m}$ of control vector $\underline{m}$ is

$$\delta\wp \geqq 0 \tag{2-36}$$

Now assume that the maximum condition for H is not satisfied during a small interval (t_a, t_b) which lies within the interval (t_0, t_f). Then for any small variation $\delta\underline{m}$ of control vector $\underline{m}$

$$H(\underline{x}, \underline{P}, \underline{m}+\delta\underline{m}, t) - H(\underline{x},\underline{P},\underline{m},t) > \epsilon \tag{2-37}$$

where t lies within the interval (t_a, t_b), and ϵ is a positive constant. A control vector $\underline{m}$ having the following properties is chosen. During the interval (t_a, t_b), $\underline{m}$ may be varied by a very small amount $\delta\underline{m}$, and outside this interval, $\underline{m}$ remains unchanged. Thus equation (2-35) becomes

$$\delta\wp = - \int_{t_a}^{t_b} \left[H(\underline{x},\underline{P}, \underline{m}+\delta\underline{m}, t) - H(\underline{x},\underline{P},\underline{m},t) \right] dt - R$$

$$= -\int_{t_a}^{t_b} \sum \frac{\partial H}{\partial m_s} \delta m_s dt - R \tag{2-38}$$

Since both $\delta \underline{m}$ and $\delta \underline{x}$ are very small, the second term at the right-hand side of equation (2-34) is an infinitesimal of higher order, which may be neglected, and then

$$R \doteq \int_{t_0}^{t_f} \sum_i \sum_j P_i \left\{ \frac{\partial [f_i(x,m+\delta m,t) - f_i(x,m,t)]}{\partial x_j} \right\} \delta x_j dt \tag{2-39}$$

A Taylor series expansion of $f_i(\underline{x}, \underline{m}+\delta\underline{m}, t)$ with respect to $\underline{m}$ results in

$$f_i(\underline{x},\underline{m}+\delta\underline{m},t) = f_i(\underline{x},\underline{m},t) + \sum_s \frac{\partial f_i(\underline{x},\underline{m},t)}{\partial m_s} \delta m_s + \epsilon^2 + \ldots \tag{2-40}$$

Since $\delta\underline{m}$ is very small, we can omit the higher order terms and then

$$R = \int_{t_0}^{t_f} \sum_i \sum_j P_i \quad \frac{\partial^2 f_i(x,m,t)}{\partial x_j \partial m_s} \delta x_j \ \delta m_s \ dt \tag{2-41}$$

In view of equation (2-17), one obtains

$$R = \int_{t_a}^{t_b} \sum\sum \frac{\partial^2 H}{\partial x_j \partial m_s} \delta x_j \ \delta m_s \ dt \tag{2-42}$$

Combining equation (2-38) and equation (2-42) yields

$$\delta\mathcal{P} = -\int_{t_a}^{t_b} \sum_s \left(\frac{\partial H}{\partial m_s} \delta m_s + \sum_j \frac{\partial^2 H}{\partial x_j \partial m_s} \delta x_j \ \delta m_s\right) dt \tag{2-43}$$

which is less than zero, since the first term of integrand is positive and the value of the second term is smaller than the first term. This implies that for this particular control vector, the Pontryagin function $\mathcal{P}$ is not minimum for any variation $\delta\underline{m}$

of the control vector $\underline{m}$.

In short, the above result points out that if the maximum condition for H is not satisfied, the minimum condition for may be violated. This proves the necessary condition.

For the proof of sufficient condition, let the process dynamics be characterized by

$$\dot{x}_i(t) = \sum_{k=1}^{n} a_{ik}(t)x_k(t) + u_i(\underline{m}) \qquad (2\text{-}44)$$

Then the Hamiltonian of the system may be expressed in the form

$$H = \sum_i \sum_k a_{ik}(t)x_k(t)P_i(t) + \sum_i u_i(\underline{m})P_i(t) \qquad (2\text{-}45)$$

Since the first term of equation (2-45) is linear in x_k and is independent of $\underline{m}$ and the second term is independent of $\underline{x}$.

$$\frac{\partial^2 H}{\partial x_j \partial m_s} = 0 \qquad (2\text{-}46)$$

So R = 0 and equation (2-35) becomes

$$\delta \mathcal{P} = - \int_{t_0}^{t_f} \left[H(\underline{x},\underline{P}, \underline{m}+\delta\underline{m}, t) - H(\underline{x},\underline{P},\underline{m},t) \right] dt \qquad (2\text{-}47)$$

Hence if the maximum condition holds for the Hamiltonian H, the integrand of equation (2-47) is nonpositive and $\delta \mathcal{P}$ is nonnegative; that is, the minimum condition for the Pontryagin function $\mathcal{P}$ is fulfilled. This proves the sufficiency condition.

3. Application of the Maximum Principle

Example. Consider the equation $\dot{x}_2 = m$, where m is a real control parameter constrained by the condition that $m = 1$. In the phase co-ordinates

$$\dot{x}_1 = x_2 \tag{2-48}$$

$$\dot{x}_2 = m \tag{2-49}$$

The problem is, "How can we get to the origin (0, 0) from an initial state x_0 in the shortest time?"

First we write the Hamiltonian function

$$H = \sum_{i=1}^{n} P_i f_i = P_1 f_1 + P_2 f_2 = P_1 \dot{x}_1 + P_2 \dot{x}_2 \tag{2-50}$$

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} = 0 \tag{2-51}$$

$$p_1 = c_1 \tag{2-52}$$

c_1 is a constant

$$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = -\frac{\partial H}{\partial \dot{x}_1} = -p_1 \tag{2-53}$$

$$p_2 = c_2 - c_1 t \tag{2-54}$$

c_2 is a constant

Taking the condition $-1 \leqq m \leqq 1$ into account, $m(t) = \operatorname{sign} p_2(t) = \operatorname{sign}(c_2 - c_1 t)$. Since $c_2 - c_1 t$ is a linear function which changes sign at most once on the interval $t_0 \leqq t \leqq t_1$, therefore for every interval $t_0 \leqq t \leqq t_1$, the optimal control m(t) is a piecewise constant function which takes on the values ± 1, and has at most t two intervals on which it is constant.

For the case m = 1. From equation (2-49) we know

$$x_2 = mt + k = t + k \tag{2-55}$$

k is a constant

From equation (2-48)

$$x_1 = \int x_2 dt = \frac{t^2}{2} + kt + 1$$

$$= \frac{1}{2}(t + k)^2 + (1 - k^2) = \frac{1}{2} x_2^2 + R \tag{2-56}$$

k, 1, R are constants.

Thus the portion of the phase trajectory for which $m = \pm 1$ is an arc of a family of parabolas shown in Fig. 2-2a.

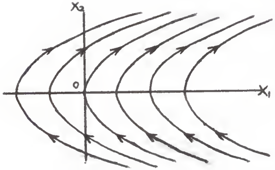

Fig. (2-2a).

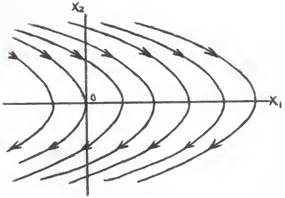

Fig. (2-2b).

For the case m = -1

$$x_2 = -t + k \tag{2-57}$$

$$x_1 = -\frac{t^2}{2} + kt + 1 = -\frac{1}{2}(-t + k') + R'$$

$$= -\frac{1}{2} x_2^2 + R' \tag{2-58}$$

The family of parabolas of equation (2-58) is shown in Fig. (2-2b). The phase points move upwards along the parabolas of equation (2-56) since $\frac{dx_2}{dt} = m = +1$; and downwards along the parabolas of equation (2-58) since $\frac{dx_2}{dt} = m = -1$.

For the case that m is initially equal to "+1", and then to "-1", the phase trajectory consists of two adjoining parabolic segments (Fig. 2-3a) if m = -1 first and m = +1 afterwards, the phase curve is shown in Fig. (2-3b).

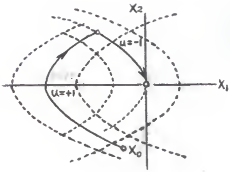

Fig. 2-3a.

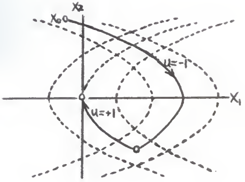

Fig. 2-3b.

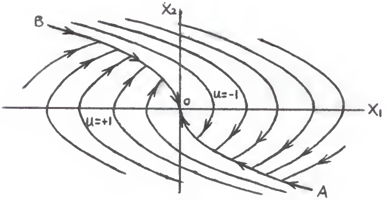

Fig. 2-4.

If we combine Figs. (2-3a) and (2-3b), we obtain Fig. (2-4). Its physical meaning is that if the phase point moves along an arc of the parabola (equation 2-58) which passes through the initial point x_0, if x_0 is above the curve AOB; and along an arc of a parabola (equation 2-56) if x_0 is below this curve. In other words, if the initial position x_0 is above the curve AOB, the phase point must move under the influence of the control $m = -1$ until it reaches the arc AO. At the instant it arrives, the value of m switches to +1 and remains at this value until the phase point reaches the origin. However, if the initial position x_0 is below AOB, m must equal +1 until the time it reaches the arc BO, and at that time the value of m changes to -1.

DYNAMIC PROGRAMMING

1. Outline

Dynamic programming, developed by the American mathematician Richard Bellman, is a simple but very powerful concept which finds applications in the solution of multistage decision problems. The basic idea is the principle of invariant embedding, according to which a very difficult or unsolvable problem is embedded into a class of simpler solvable problems, so that a solution can be obtained. Numerous applications of dynamic programming techniques are possible, but here we are interested in its application to optimal control problems.

In general, multistage decision problems are best solved by

means of the functional equation approach, and the functional equation describing a multistage decision process can readily be derived by invoking "the principle of optimality", which states: "An optimal policy has the property that whatever the initial state and initial decision arc, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

The principle of optimality, which motivates the basic properties of optimal control strategies, is based upon the fundamental concept of invariant embedding. This concept implies that to solve a specific optimum decision problem, the original problem is embedded within a family of similar problems which are easier to solve. For multistage decision processes, this will allow the replacement of the original multistage optimization problem by the problem of solving a sequence of single-stage decision processes, which are simpler to handle.

Let $\underline{x}$ be k^{th} state vector characterizing a physical system at any time. If the state of the physical system is transferred from $\underline{x}_1$ into $\underline{x}_2$ by the transformation

$$\underline{x}_2 = g(\underline{x}_1, m_1) \tag{3-1}$$

For a single-stage decision process, an output or return yields,

$$R_1 = r(x_1, m_1) \tag{3-2}$$

The problem is to choose a decision m_1 so as to maximize the return. The maximum return is given by

$$f_1(\underline{x}_1) = \max_{m} r(\underline{x}_1, m_1) \tag{3-3}$$

In a two-stage decision process, if the state of the physical system is first transformed from $\underline{x}_1$ into $\underline{x}_2$ by

equation (3-1) and is then transformed from $\underline{x}_2$ into $\underline{x}_3$ by the transformation.

$$\underline{x}_3 = g(\underline{x}_2, m_2) \tag{3-4}$$

This sequence of operations results in a total return.

$$R_2 = r(\underline{x}_1, m_1) + r(\underline{x}_2, m_2) \tag{3-5}$$

Then the optimum design problem is to choose a sequence of allowable decisions m_1 and m_2 so as to maximize the total return. The maximum return is given by

$$f_2(\underline{x}_2) = \max_{m_1, m_2} \left\{ r(\underline{x}_1, m_1) + r(\underline{x}_2, m_2) \right\} \tag{3-6}$$

In general, for an N-stage decision process, the problem is to choose an N-stage policy

$$\left\{ m_1, m_2, m_3, \ldots, m_N \right\}$$

so as to maximize the total return

$$R_N = \sum_{j=1}^{N} r(\underline{x}_j, m_j) \tag{3-7}$$

The maximum return of the N-stage process is given by

$$f_N(\underline{x}_1) = \max_{m_j} \left\{ \sum_{j=1}^{N} r(\underline{x}_j, m_j) \right\} \tag{3-8}$$

It is not expected to obtain the solution of N simultaneous equation by zeroing the partial derivatives of the quantity in the braces with respect to m_j, $j = 1, 2, \ldots, n$, but the problem can be solved by using the principle of optimality. If

$$R_N = r(\underline{x}_1, \underline{m}_1) + f_{N-1}\left[g(\underline{x}_1, m_1)\right] \tag{3-9}$$

then the maximum return is given by

$$f_N(\underline{x}_1) = \max_{m_1} \left\{ r(\underline{x}_1, m_1) + f_{N-1} \left[g(\underline{x}_1, m_1)\right] \right\} \tag{3-10}$$

Clearly, by applying the principle of optimality, the N-stage

decision process is reduced to a sequence of N single-stage decision processes, thus enabling this optimization problem to be solved in a systematic, iterative manner.

We should pay attention to the fact that any multistage decision process in realistic situations is usually time dependent and stochastic. In other words, it is unreasonable to assume a multistage decision process to be independent of time or to assume that the process is deterministic in nature. We shall consider these realistic conditions in the following problems.

2. Basic Principle and Application

We will now discuss the application of dynamic programming technique to the three basic types of problems--minimum-integral control processes, terminal-control processes, and minimum-time processes.

1). Minimum-integral Control Processes. Consider an n^{th}-order control process characterized by the vector differential equation:

$$\underline{\dot{x}}(t) = \underline{g}(\underline{x}, \underline{m}, t) \qquad (3\text{-}11)$$

where $\underline{x}$ = an n-vector representing state of process

$\underline{m}$ = an r-vector denoting control signals

$\underline{g}$ = a differentiable vector function of the arguments $\underline{x}$, $\underline{m}$, t.

The initial conditions are given by

$$\underline{x}(t_0) = \underline{x}_0 \qquad (3\text{-}12)$$

Determine the optimal control vector $\underline{m}$ which minimizes the integral criterion function

$$I(m) = \int_{t_0}^{t_f} F(\underline{x}, \underline{m}, t)dt \tag{3-13}$$

The integrand $F(\underline{x}, \underline{m}, t)$ is a differentiable scalar function of the state vector control vector and time.

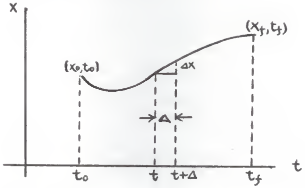

Fig. 3-1. Optimum trajectory.

In order to apply the functional equation technique of dynamic programming, this optimization problem is embedded within the wider problem of minimizing

$$\int_{t}^{t_f} F(\underline{x}, \underline{m}, t)dt$$

We write

$$f(\underline{x}, t) = \min_{\underline{m}} \int_{t}^{t_f} F(\underline{x}, \underline{m}, t)dt \tag{3-14}$$

where t ranges over the interval (t_0, t_f), the minimum is taken over all $\underline{m}$, and $f(\underline{x}, t_0) = f(\underline{x}_0)$ at $t = t_0$. Application of the principle of optimality reduces equation (3-14) to the functional equation.

$$f(\underline{x},\ t) = \min_{\underline{m}} \left[\int_t^{t+\Delta} F(\underline{x},\underline{m},\tau)d\tau + f(\underline{x} + \dot{\underline{x}}\Delta,\ t + \Delta) \right] \tag{3-15}$$

By integration and Taylor series expansion one obtains

$$f(\underline{x},t) = \min_{\underline{m}}\ F(\underline{x},\underline{m},t)\Delta + f(\underline{x},t) + \dot{\underline{x}}\frac{f}{\underline{x}}\Delta + \frac{f}{t}\Delta + \epsilon(\Delta) \tag{3-16}$$

If Δ approaches zero, equation (3-16) becomes

$$-\frac{\partial f}{\partial t} = \min_{\underline{m}}\ F(\underline{x},\ \underline{m},\ t) + \frac{\partial f}{\partial \underline{x}} \underline{g}(\underline{x},\ \underline{m},\ t) \tag{3-17}$$

From equation (3-17) the following two equations result:

$$\frac{\partial F}{\partial \underline{m}} + \frac{\partial f}{\partial \underline{x}}\frac{\partial \underline{g}}{\partial \underline{m}} = 0 \tag{3-18}$$

$$F + \frac{\partial f}{\partial \underline{x}} \underline{g}(\underline{x},\ \underline{m},\ t) + \frac{\partial f}{\partial t} = 0 \tag{3-19}$$

Solving for $\frac{\partial f}{\partial \underline{x}}$ and $\frac{\partial f}{\partial t}$ from equation (3-18) and equation (3-19), respectively, yields

$$\frac{\partial f}{\partial \underline{x}} = -\frac{\partial F/\partial \underline{m}}{\partial \underline{g}/\partial \underline{m}} = p(\underline{x},\ \underline{m},\ t) \tag{3-20}$$

$$\frac{\partial f}{\partial t} = -F(\underline{x},\ \underline{m},\ t) - \frac{\partial f}{\partial \underline{x}} \underline{g}(\underline{x},\ \underline{m},\ t)$$

$$= -F(\underline{x},\ \underline{m},\ t) + \frac{\partial F/\partial \underline{m}}{\partial \underline{g}/\partial \underline{m}} \underline{g}(\underline{x},\ \underline{m},\ t) = Q(\underline{x},\ \underline{m},\ t) \tag{3-21}$$

By partial differentiation of equation (3-20) with respect to t and partial differentiation of equation (3-21) with respect to $\underline{x}$, we obtain

$$\frac{\partial^2 f}{\partial \underline{x}\partial t} = \frac{\partial P}{\partial t} + \frac{\partial P}{\partial \underline{m}}\frac{\partial \underline{m}}{\partial t} + \frac{\partial P}{\partial \underline{x}}\frac{\partial \underline{x}}{\partial t} \tag{3-22}$$

$$\frac{\partial^2 f}{\partial \underline{x}\partial t} = \frac{\partial Q}{\partial \underline{x}} + \frac{\partial Q}{\partial \underline{m}}\frac{\partial \underline{m}}{\partial \underline{x}} \tag{3-23}$$

Combining equation (3-22) and equation (3-23),

$$\frac{\partial P}{\partial t} + \frac{\partial P}{\partial \underline{m}}\frac{\partial \underline{m}}{\partial t} + \frac{\partial P}{\partial \underline{x}}\frac{\partial \underline{x}}{\partial t} = \frac{\partial Q}{\partial \underline{x}} + \frac{\partial Q}{\partial \underline{m}}\frac{\partial \underline{m}}{\partial \underline{x}} \tag{3-24}$$

From equation (3-24), the optimum control $\underline{m}$ can be determined.

In the foregoing study, we assume that no constraint is imposed upon control signals and state variables. In practice, because of the physical limitations of controlling devices, physical constraints on control signal or state variables must be taken into account in the problems of optimum controls. If the constraints are not considered, the design may lead to a system demanding excessively large control signals, which is unrealistic and impracticable. Commonly encountered constraints on control signals may be described as

a) integral constraint

$$\int_{t_0}^{t_f} \underline{H}(\underline{m})dt \leq \underline{c} \tag{3-25}$$

b) amplitude saturation

$$a_i \leq m_i(t) \leq b_i \tag{3-26}$$
$$t_0 \leq t \leq t_f$$

where a_i, b_i = constants

$\underline{c}$ = a constant vector

$\underline{H}(m)$ = a vector function of the control signals.

a) The first kind of constraints can be handled by the

Lagrange multiplier method. (This was mentioned in part 2, calculus of variations.) The optimization problem is transformed into the problem of minimizing the synthetic function.

$$I_1(\underline{m}) = I(\underline{m}) + \underline{\lambda}' \int_{t_0}^{t_f} \underline{H}(\underline{m})dt \qquad (3\text{-}27)$$

where $I(\underline{m})$ is the specified performance index to be minimized. $\underline{\lambda}'$ is the transpose vector Lagrange multiplier. After the minimization of the synthetic function is achieved, the desired control vector results as a function of the vector Lagrange multiplier $\underline{\lambda}$. Substitution of the vector $\underline{m}(\underline{\lambda})$ into equation (3-25) leads to r equations which can be solved for the elements λ_i of the vector Lagrange multiplier.

b) When constraints of the second type are considered, the analytical solutions of partial differential equations and the Euler-Lagrange equation which will appear in the optimum problems are extremely difficult to obtain. To circumvent these difficulties, the following functional equations will be used:

$$\begin{aligned} f(\underline{x}, t) &= \min_{\underline{m}} \left[\int_t^{t+\Delta} F(\underline{x}, \underline{m}, t)dt + f(\underline{x}, + \dot{\underline{x}}\Delta , t + \Delta) \right] \\ &= \min_{\underline{m}} \left[F(\underline{x}, \underline{m}, t) \quad + f(\underline{x} + \underline{g}\Delta , t + \Delta) \right] \qquad (3\text{-}28) \end{aligned}$$

where the control signal satisfies the constraint $|m| \leqq M$, and Δ is a predetermined small interval of time. An N-stage computational process yields

$$f_{N-j}(\underline{x},t) = \min_{\underline{m}} \; F(\underline{x},\underline{m},t)\Delta + f_{N-(j+1)}(\underline{x} + \underline{g}\Delta , t + \Delta) \qquad (3\text{-}29)$$

with $t = t_0 + j \quad , \; j = 0, 1, 2, \ldots, N$

$$t_f = t_0 + N\Delta$$

For j=0 $\quad f_N(\underline{x}, t_0) = \min_{\underline{m}} \left[F(\underline{x}, \underline{m}, t_0) + f_{N-1}(\underline{x}+\underline{g}\Delta, \ t_0+\Delta) \right]$

$$(3\text{-}30)$$

$$j=N \quad f_0(\underline{x},\ t_f) = 0 \tag{3-31}$$

$$j=n-1 \quad f_1(\underline{x},\ t_f-\Delta) = \min_{\underline{m}} F(\underline{x},\underline{m},\ t_f-\Delta) \ + f_0(x, t_f)$$

$$= \min_{\underline{m}} F(\underline{x},\ \underline{m},\ t_f - \Delta)\Delta \tag{3-32}$$

and

$$f_2(\underline{x},\ t_f - 2\Delta) = \min_{\underline{m}} F(\underline{x},\underline{m},\ t-2\Delta)\Delta + f_1(\underline{x},\ t_f - \Delta) \tag{3-33}$$

From equation (3-32) and equation (3-33), the values of $f_1(\underline{x},\ t_f - \Delta)$ and the corresponding m_1, $f_2(\underline{x},\ t_f - 2\Delta)$ and the corresponding m_2 are determined for successive values of $\underline{x}$. The values of $f_1(\underline{x},\ t_f - \Delta)$ may be obtained from the foregoing either directly or by interpolation or extrapolation. By this method, the optimum control signal m which minimizes the specified integral criterion function can be ascertained.

2) Terminal-control Processes. Consider the n^{th}-order control process characterized by the differential equation

$$\dot{\underline{x}}(t) = A\underline{x}(t) + D\underline{m}(t) \tag{3-34}$$

with the initial conditions given by $\underline{x}(0) = \underline{x}_0$. In equation (3-34) $\underline{x}$ is an n-vector representing the state of the control process, $\underline{m}$ is an r-vector representing the control signals subjected to the following constraints during the interval $0 \leqq t \leqq T$:

$$|m_k| \leqq M_k \tag{3-35}$$

$$k = 1, 2, \ldots, r$$

$$\int_0^T \underline{H}(m)\,dt \leqq c \tag{3-36}$$

A and D are the coefficient and the driving matrices, respectively. The optimum control vector $\underline{m}(t)$ is to be determined so as to minimize the criterion function.

$$I(\underline{m}) = G\left[x_1(T),\ x_2(T),\ .\ .\ .,\ x_1(T)\right] \qquad (3\text{-}37)$$

It has been found that the solution to equation (3-34) is

$$\underline{x}(t) = \underline{z}(t) + \int_0^t \underline{w}(t - \tau)\ \underline{m}(\tau)\ d\tau \qquad (3\text{-}38)$$

where $\underline{z}(t) = \phi(t)\ X(0)$ = complementary solution and

$\phi(t) = e^{(t-t_0)A}$

$\underline{w}(t - \tau)\underline{m}(\tau)d\tau$ = particular solution

$\underline{w}(t - \tau) = \phi(t - \tau)D = e^{(t-\tau)}D$

In terms of vector components, the state variables of the control process are given by

$$x_i(t) = z_i(t) + \int_0^t \left[\sum_{k=1}^{r} w_{ik}(t - \tau)m_k(\tau)\right] d\tau \qquad (3\text{-}39)$$

$$i = 1,\ 2,\ .\ .\ .,\ n$$

The terminal control problem may now be restated as determination of the optimum control vector $\underline{m}(t)$ that minimizes the function

$$I(\underline{m}) = G\left\{z_1(T) + \int_0^T \sum_{k=1}^{r} w_{ik}(t - \tau)m_k(\tau)d\tau\ .\ .\ .\ .\right.$$

$$\left. z_1(T) + \int_0^T \sum_{k=1}^{r} w_{1k}(t - \tau)m_k(\tau)d\tau\right] \qquad (3\text{-}40)$$

subject to constraints given in equation (3-35) and equation (3-36).

Application of the Lagrange multiplier converts this

optimization problem into the minimization of the synthetic function

$$I_1(m) = G + \lambda \int_0^T H(\underline{m})\,d\tau \tag{3-41}$$

with respect to $\underline{m}$, which is now constrained only by equation (3-35). Let the minimum value of I_1 be denoted by $f(z_1, z_2, \ldots, z_1, T)$. Then

$$f(z_1, z_2, \ldots, z_1, T) = \min_{m_k} \Big\{ G \Big[z_1(T) + \int_0^T \sum_k w_{ik}(T - \tau) m_k(\tau)\,d\tau, \ldots, z_1(T) + \int_0^T \sum_k w_{1k}(T-\tau) m_k(\tau)\,d\tau \Big] + \lambda \int_0^T H(m_1, m_2, \ldots, m_r)\,d\tau \Big\} \tag{3-42}$$

which may be rewritten as

$$f(z_1, z_2, \ldots, z_1, T) = \min_{m_k} \Big\{ G \Big[z_1(T) + \int_0^\Delta \sum_k w_{ik}(T-\tau) m_k(\tau)\,d\tau + \int_\Delta^T \sum_k w_{ik}(T-\tau) m_k(\tau)\,d\tau + \ldots z_1(T) + \int_0^\Delta \sum_k w_{1k}(T-\tau) m_k(\tau)\,d\tau + \int_\Delta^T \sum_k w_{1k}(T-\tau) m_k(\tau)\,d\tau \Big] + \lambda \int_0^\Delta H(m_1, m_2, \ldots, m_r)\,d\tau + \int_\Delta^T H(m_1, m_2, \ldots, m_r)\,d\tau \Big\} \tag{3-43}$$

where Δ is a very small time interval. By changes of limits,

$$f(z_1, z_2, \ldots, z_1, T) = \min_{m_k} \Big\{ G \Big[z_1(T) + \int_0^\Delta \sum_k w_{ik}(T-\tau) m_k(\tau)\,d\tau$$

$$+ \int_0^{T-\Delta} \sum_k w_{ik}(T-\tau-\Delta)m_k(\tau+\Delta)d\tau \ \ldots \ z_1(T)$$

$$+ \int_0^{\Delta} \sum_k w_{1k}(T-\tau)m_k(\tau)d\tau + \int_0^{T-\Delta}{}_k \ w_{1k}(T-\tau-\Delta)m_k(\tau+\Delta)d\tau\Big]$$

$$+ \lambda \int_0^{\Delta} H(m_1,m_2,\ldots,m_r)d\tau + \lambda \int_0^{T-\Delta} H(m_1,m_2,\ldots,m_r)d\tau\Big\} \tag{3-44}$$

By the principle of optimality, the functional equation for terminal control problem is found to be

$$f(z_1,z_2, \ldots, z_1,T) = \min_{m_k}\Big\{ f\Big[z_1 + \int_0^{\Delta} \sum_k w_{ik}(T-\tau)m_k(\tau)d\tau, \ldots z_1$$

$$+ \int_0^{\Delta} \sum_k w_{1k}(T-\tau)m_k(\tau)d\tau; \quad T-\Delta\Big] + \lambda \int_0^{\Delta} H(m_1,m_2,\ldots,m_r)d\tau\Big\} \tag{3-45}$$

where the minimum is taken over all m_k defined over the interval $(0, \Delta)$ and satisfying constraints given in equation (3-35). An analytical solution of the terminal control problem is not easy to derive, but by approximating the following recurrence relationship we may obtain the solution.

$$f(z_1,z_2,\ldots,z_1,T) = \min_{m_k}\Big\{ f\Big[z_1 + \Delta\sum_k w_{ik}(T)m_k, \ldots, z_1$$

$$\Delta\sum_k w_{1k}(T)m_k; \quad T-\Delta\Big] + \lambda H(m_1,m_2,\ldots,m_r)\Big\} \tag{3-46}$$

with

$$f(z_1,z_2,\ldots,z_1; 0) = G\big[z_1(0), z_2(0), \ldots, z_1(0)\big] \tag{3-47}$$

3) <u>Minimum-time</u> <u>Control</u> <u>Process</u>. Consider the nonlinear control process characterized by the vector differential equation

$$\underline{\dot{x}} = \underline{g}(\underline{x}, \underline{m}, t) \tag{3-48}$$

where $\underline{x}$ is n-state vector and $\underline{m}$ is r-vector. Determine the optimum-control strategy which will transform the process from a given initial state

$$\underline{x}(t_0) = \underline{x}_0 \tag{3-49}$$

to the desired final state.

$$\underline{x}(t_f) = \underline{x}_{tf} \tag{3-50}$$

in minimum time, and the final time t_f is unspecified.

Let the function of the minimum time to transform the process from the state $\underline{x}(t)$ to the desired final state $\underline{x}(t_f)$ with the control vector optimally chosen be

$$f(\underline{x}, t) = \min_{\underline{m}} \left\{ t_f - t_0 \right\} \tag{3-51}$$

On the basis of the definition of $f(\underline{x}, t)$, it follows that at the end of the optimal trajectory

$$\left.\frac{\partial f}{\partial t}\right|_{t=t_f} = 0 \tag{3-52}$$

and

$$\left.\frac{\partial f}{\partial x_i}\right|_{t=t_f} = 0 \tag{3-53}$$

By the principle of optimality, the minimum time is given by the functional equation.

$$f(\underline{x}, t) \min_{\underline{m}} \left\{ \Delta + f(\underline{x} + \underline{\dot{x}}\Delta, t + \Delta) \right\} \tag{3-54}$$

Expanding $f(\underline{x} + \underline{\dot{x}}\Delta, t + \Delta)$ into Taylor series and simplifying as before, we obtain

$$\min_{\underline{m}} \left\{ \Delta + \frac{\partial f}{\partial \underline{x}} \underline{g}(\underline{x}, \underline{m}, t)\Delta + \frac{\partial f}{\partial t} \Delta + \epsilon(\Delta) \right\} = 0 \tag{3-55}$$

If Δ approaches zero as a limit, equation (3-55) becomes

$$\min_{\underline{m}} \left[\frac{\partial f}{\partial \underline{x}} \underline{g}(\underline{x}, \underline{m}, t) + \frac{\partial f}{\partial t} \right] = -1 \qquad (3\text{-}56)$$

From equation (3-56) and from arguments similar to those for deriving equations (3-18) and (3-19), obtain the following two equations:

$$\frac{\partial f}{\partial \underline{x}} \frac{\partial \underline{g}(\underline{x}, \underline{m}, t)}{\partial \underline{m}} = 0 \qquad (3\text{-}57)$$

$$\frac{\partial f}{\partial \underline{x}} \underline{g}(\underline{x}, \underline{m}, t) + \frac{\partial f}{\partial t} + 1 = 0 \qquad (3\text{-}58)$$

Furthermore, if the function $\underline{g}$ is time-independent, equation (3-58) can be reduced as

$$\frac{\partial f}{\partial \underline{x}} \underline{g}(\underline{x}, \underline{m}) + 1 = 0 \qquad (3\text{-}59)$$

when the partial derivative $\frac{\partial f}{\partial \underline{x}}$ is known at a point, equation (3-57) and equation (3-59) can be solved for the optimal control vector $\underline{m}$.

SUMMARY

Among the many optimum-control problems, three basic types are of fundamental importance. They are the minimum-time control problem, the terminal-control problem, and the minimum-integral control problem. Based on the three kinds of problems, we have discussed the optimal-control system by the calculus of variations, the maximum principle, and dynamic programming.
In discussing the calculus of variations, the minimum-integral control problem is classified as the Lagrange type, the minimum-time control and the terminal-control problems are classified as Mayer problem.

The optimum design of control systems by the calculus of variations generally leads to a two-point boundary value problem. Analytical solutions for such problems are possible only in special cases. In view of the fact that the resulting Euler-Lagrange differential equations are usually nonlinear, numerical trial-and-error techniques must be resorted to. These techniques employ an initial value for the missing initial condition and integrating numerically the Euler-Lagrange equations and constraining equations. This work belongs to applied mathematics and numerical analysis, and was not discussed here. The difficulty in solving a two-point boundary value problem makes the classical calculus of variations less attractive in the design of an optimal-control system. Further, the variational calculus approach is generally limited to systems subject to control signals with unrestricted bounds.

The maximum principle of Pontryagin provides an elegant method of obtaining an optimal solution for very general dynamical processes. It treats the optimization problem of maximizing or minimizing a function subject to certain constraints. In general, a new state variable x_{n+1} is introduced to convert the optimum control problem to the optimization of this new co-ordinate, and the Pontryagin function $\Phi = \underline{b}'\underline{x}(t_f)$ subject to certain constraints is used for this new co-ordinate.

In general, the maximum principle provides a necessary condition for system optimization. However, if the control process is linear and subject to an additive control function, it provides the necessary and sufficient condition for optimum control. Although the application of the maximum principle is not restricted to systems with unbounded control signals, it is subject to the same difficult two-point boundary value problem in the variational calculus.

The basic theory of dynamic programming is the principle of optimality and the functional equation approach. Following the formal analysis, the optimal control problem can be reduced to the determination of the solution of the Hamilton-Jacobi equation. The functional equation approach of dynamic programming provides a way of obtaining the computational solution of optimization problem which does not depend upon the solution of the partial differential equation, thus circumventing difficulties with the two-point boundary value problem. The principle of optimality is applied to the derivation of the partial differential equations describing the optimal control signals.

Constraints on control signals are considered in the optimum design. For control processes of moderate complexity, a solution to the partial differential equation is generally difficult to derive, and resort is often made to numerical analysis through the functional equation approach. The constraint on the control signal defines a finite range of possible values, and this makes the computation easier.

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A REVIEW OF OPTIMAL CONTROL SYSTEM ANALYSIS

by

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AN ABSTRACT OF A MASTER'S REPORT

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Optimal control problems are viewed as variational problems. Three basic variational methods are discussed for maximizing or minimizing a functional over a function space. They are:

1. The calculus of variations
2. The maximum principle
3. Dynamic programming.

Three basic problems in optimal control systems are stated as typical problems. They are:

1. The minimum-time control problem
2. The terminal-control problem
3. The minimum-integral control problem.

In all cases the goal is to find the optimum control law or sequence such that the given function of the performance indices is maximized or minimized. In realistic and practical situations, physical constraint on control signals or state variables must be taken into account and these make optimum control problems more complicated.